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Equivalent mass, stiffness, and loading for off-centre missile impact evaluations

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ABSTRACT

For evaluation of the structures against missile impact loads, it is usually desirable to idealize the structure with a single degree of freedom (SDOF) system. It is therefore convenient to introduce certain transformation factors to convert the mass and existing loads of the structure to equivalent mass, and load for the equivalent SDOF system.

The transformation factors are a function of the deflected shape of the structure under dynamic impacts. For impacts at the centre of common elements, close approximation of these factors are available in the form of tabulated data in the literature. However, they are not readily available for impacts occurring at a point away from the centre where these factors turn out to be somewhat greater than the values for centre impacts, and therefore, the published tabulated data are not applicable. This paper investigates the transformation factors for non-central impacts on beams and slabs. For the beam, the transformation factors are obtained based on different deflection shape functions. The first function is based on the Euler beam theory, neglecting the shear deflection. The second function considers the shear deflection based on Castigliano's theorem. Lastly, the shape function is obtained by Finite Element Analysis (FEA) where the beam is represented with shell elements. The results obtained based on FEA are considered the most accurate and are used as reference for comparison. Similarly, a slab is investigated by FEA. The results obtained are tabulated so that they can be used for design against missile impacts. A discussion is provided for selecting the most applicable deflected shape function for a given impact location.

INTRODUCTION

Representation of the actual structure and loading with an equivalent single degree of freedom (SDOF) system significantly simplifies the dynamic analysis of the structure since one only needs to deal with the analysis of a single degree of freedom system for which, a closed form solution may be available or simple hand calculation may provide reasonable approximation of the response. The equivalent SDOF system is usually selected so that the deflection of the equivalent SDOF mass is the same as that for the real structure at the location of interest, i.e. the location that is being impacted by say, a tornado generated missile or an accidental heavy load drop.

In order to idealize the real structure with the equivalent SDOF, it is necessary to evaluate the dynamic characteristics of the structure. These parameters are the mass, stiffness and effect of non-impact existing load, namely m_e, k_e and F_e. In order to calculate these parameters, it is convenient to determine transformation factors which can be used to convert the total mass and existing loading of the real system to those of the equivalent SDOF. When the total load and mass of the structure are multiplied by the corresponding transformation factors, one can obtain those parameters for the equivalent SDOF system.

The following provides a summary of the mathematical procedure for the calculation of transformation factors for beams and slabs.

METHODOLOGY

The original derivation of the method is given by Timoshenko (1937) and is carried out on a single degree of freedom (SDOF) mass-spring system replacing the actual structure.

The dynamic parameters of the equivalent system are evaluated on the basis of an assumed shape of the actual structure. This shape will be taken to be the same as that resulting from the static application of the dynamic loads at point of impact.

Mass transformation factor:

The equivalent mass of a mode, or in the present case, of the equivalent SDOF for a line structure (such as a beam) with continuous mass, is given by

$$M_e = \int m_L \phi_n(\mathrm{x})^2 dx \qquad (1)$$

Similarly, for a two dimensional structure such as a plate or a slab, the equivalent mass is given by

$$M_e = \int m_A \phi_n(\mathrm{x}, \mathrm{y})^2 dxdy \qquad (2)$$

In Equations (1) and (2), ϕ_n is the assumed normalized shape function on which the equivalent system is based, and m_L and m_A represent the mass per unit length and mass per unit area, respectively.

The mass factor is defined as the ratio of the equivalent mass to the total mass of the structure.

$$K_M = \frac{M_e}{M_t} \qquad (3)$$

For example, in the case of beam with uniform mass per unit length m, $M_t = m.L$, where L is the span.

Load transformation factor:

Similarly, the equivalent load on the idealized system with a distributed load p for a line structure is given by

$$F_e = \int p(x)\phi_n(\mathrm{x})dx \tag{4}$$

Similarly, for a two dimensional structure with a distributed load p, the equivalent load is given by

$$F_e = \int p(x,y)\phi_n(\mathrm{x},\mathrm{y})dxdy \tag{5}$$

The load factor K_L is defined as the ratio of the equivalent load to total load.

$$K_L = \frac{F_e}{F_t} \tag{6}$$

For example, for the case of a beam with uniform load per unit length p, $F_t = p.L$ where L is the beam length.

As can be seen in equations (1), (2), (4) and (5) above, the transformation factors are a function of the deflected shape of the structure. The shape functions are generally based on the static deflected shape corresponding to the particular load distribution. For beam elements without the consideration of the shear deflections, the shape functions for different loading and boundary conditions are given in various literatures including the Steel Design Manual (2005). The three shape functions for three common boundary conditions shown in Figures 2a, 2c, and 2d are given below for convenience:

- Beam simply supported on both ends - Concentrated load P at distance a from the support

$$\phi_b(\mathrm{x}) = \frac{P(L-a)x}{6EI}(L^2-(L-a)^2-x^2) \qquad when\ x \le a \tag{7}$$

- Beams fixed at both ends - Concentrated load P at distance a from the support

$$\phi_b(\mathrm{x}) = \frac{P(L-a)^2x^2}{6EIL^3}(3aL-3ax-(L-a)x) \qquad when\ x \le a \tag{8}$$

- Beams fixed at one end and simply supported at the other end - Concentrated load P at distance a from the support

$$\phi_b(\mathrm{x}) = \frac{P(L-a)^2x}{12EIL^3}(3aL^2-2Lx^2-ax^2) \qquad when\ x \le a \tag{9}$$
$$\phi_b(\mathrm{x}) = \frac{Pa(L-x)^2}{12EIL^3}(3L^2x-a^2x-2a^2L) \qquad when\ x > a \tag{10}$$

In the Equations (7) through (10), E is the beam material elastic modulus, I is the beam section moment of inertia, and a is the distance from the location of the equivalent SDOF system to the beam support.

The shear deflection of the beam may become significant when the load is applied close to the beam supports. The deflection due to shear can be found by classical methods such as unit loads or by Castigliano's theorem. Application of these theories to a beam with a concentrated load yields the following formula for shear deflection of the beam:

$$\phi_v(\mathrm{x}) = \beta \frac{P(L-a)}{AG} . \frac{x}{L} \qquad when\ x \leq a \tag{11}$$

$$\phi_v(\mathrm{x}) = \beta \frac{Pa}{AG} . \frac{L-x}{L} \qquad when\ x > a \tag{12}$$

where β is a function of the beam section properties, A is the beam cross section area, and G is the beam material shear modulus.

According to Young and Budynas (2002), approximate results may be obtained for I beams using $\beta=1$.

The total deflection of the beam at any points along its length can be obtained by summing the shear and bending deflections at that point:

$$\phi(\mathrm{x}) = \phi_b(\mathrm{x}) + \phi_v(\mathrm{x}) \tag{13}$$

The normalized shape function $\phi_n(\mathrm{x})$ is then defined as:

$$\phi_n(\mathrm{x}) = \frac{\phi(\mathrm{x})}{\phi(\mathrm{a})} \tag{14}$$

Stiffness transformation factor:

The stiffness of an element is the internal force tending to restore the element to its unloaded static position. It is numerically equal to the total load of the same distribution which would cause a unit deflection at the point where the deflection is equal to that of the equivalent system. Based on this definition, when the location of the equivalent SDOF system is the same as the location of the impact load, the stiffness transformation factor becomes one.

It should be noted that the shape functions given in Equations (7) through (12) can be used so long as the deflected shape of the beam is governed by the concentrated impact load rather than the distributed existing loads.

Various authors have previously used reasonable approximations of the deflected shape function $\phi(x)$ to establish the equivalent SDOF parameters for simple structures. For example, Whitney et al. (1955), Williamson and Alvy (1957) and Biggs (1964) provide values of transformation factors for beams and slabs with various boundary conditions. However, the previous studies provide the transformation factors at the centre of the beam or slabs. Since the location of the impact for loading conditions such as tornado generated missile impacts and accidental heavy load drop may not be limited to centre of impacted element; most often, the design engineer needs to evaluate the impact at other (off-centre) locations as well. In this paper, the works of the previous authors have been extended to obtain the transformation factors for impact locations other than the centre location. This paper investigates two structural elements, namely beam and slab.

CASE STUDIES

A W16x67 I beam is used in this paper for the case study of the beams. The properties of the beam are shown in Table 1 below:

Table 1: Beam details used for the case study

beam section	W16x67	web thickness (t_w)	0.395 in.
total length (L)	32 ft.	cross section area (A)	19.6 in^2
beam height (d)	16.3 in.	section moment of inertia	954 in^4
flange width (T)	6.7 in.	material elastic modulus	29000 ksi
flange thickness (t_f)	0.665 in.	Poisson's ratio	0.3

Since the accuracy of the transformation factors is only as good as the accuracy in calculating the deflection shape function, a sensitivity study is performed based on various deflection shape functions in beams. Three shape functions are evaluated in this study. The first function only considers the flexural deflection of the beam per Equations (7) through (10). The second shape function considers the shear deflection of the beam (Equations (13)) as well as the bending deflection. Lastly, the shape function is obtained by Finite Element Analysis (FEA) where the beam is represented with thick shell elements in SAP2000 program as shown in Figure 1. The shape function obtained based on FEA for each load location is considered the most accurate and is used as reference for comparison. The beam loading and boundary conditions used in this case study are shown in Figure 2. The transformation factors calculated by the shape functions described above at several beam locations are shown in Table 3 for simply supported beam, Table 4 for fixed beam and Table 5 for a beam fixed at one end and simply supported at the other end.

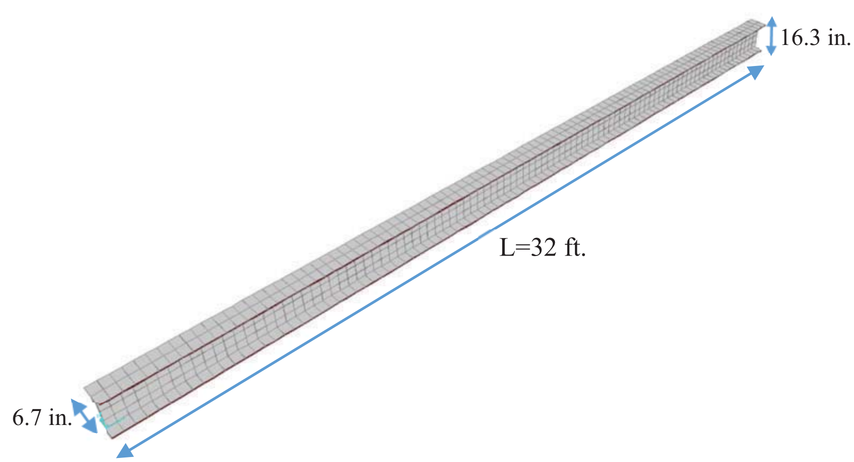

Figure 1: Schematic of the beam FE model showing the overall dimensions

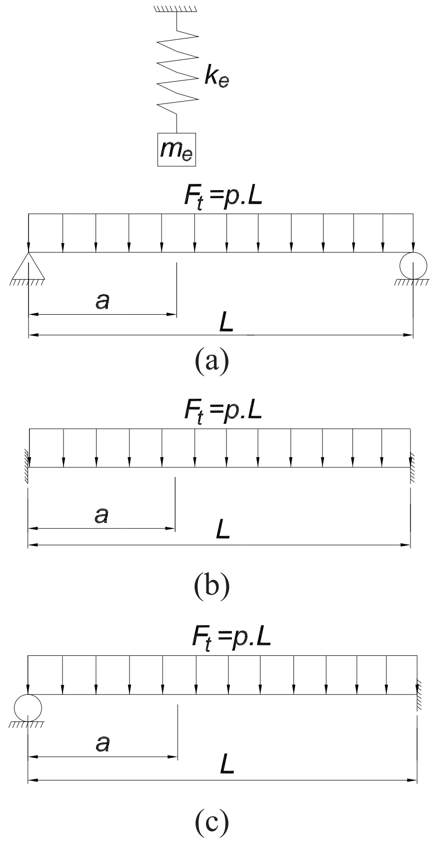

Figure 2: Boundary conditions of the beams considered in this study

Similarly, a slab with a unit aspect ratio as shown in Figure 3 is investigated by FEA using thick shell elements in SAP2000 program for various equivalent SDOF locations. The properties of the slab are shown in Table 2 below:

Table 2: Slab characteristics used for this case study

slab material	concrete	elastic modulus	3605 ksi
slab thickness	12 in.	shear modulus	1502 ksi
slab total width	32 ft.	Poisson's ratio	0.2

The evaluation of the slab for other aspect ratios is not deemed necessary since the results can be extrapolated for other aspect ratios (a slab with aspect ratio of two is essentially a one-way slab and can therefore be treated as a beam.). The shape functions used for the slab are directly extracted from the FEA results. The slab boundary conditions used in this case study are shown in Figure 4. The transformation factors at different slab locations are shown in Table 6.

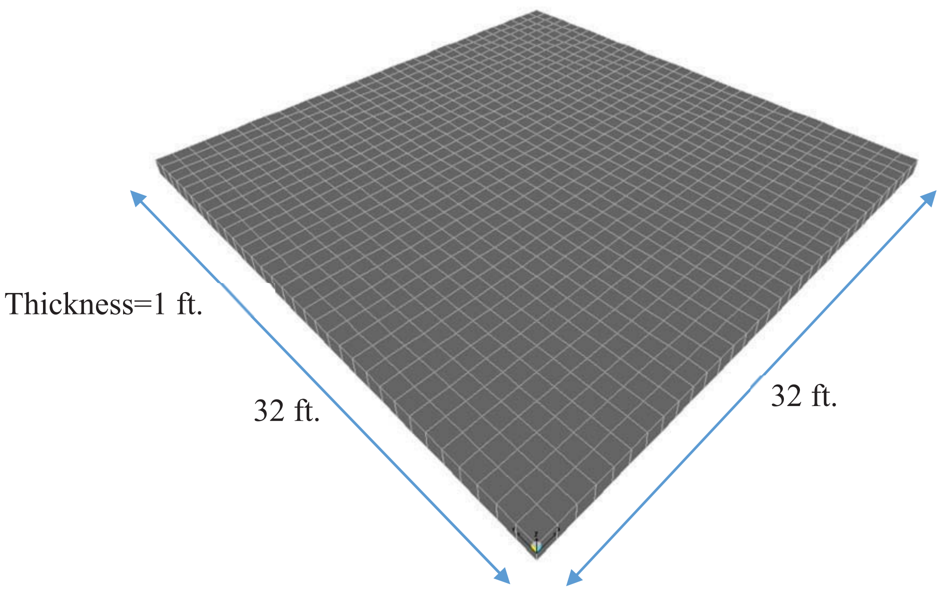

Figure 3: Schematic of the slab FE model showing the overall dimensions

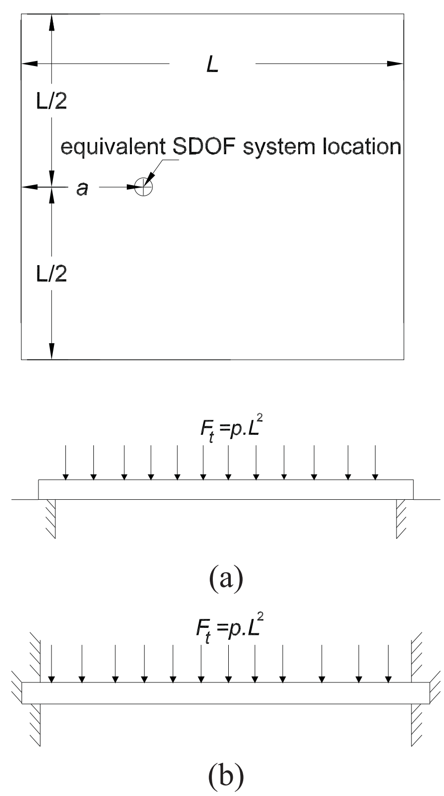

Figure 4: Boundary conditions of the slab/plate considered in this study

Comparison of the results shown in Tables 3 through 5 shows that the inclusion of shear deflection in the estimation of the transformation factors when the load application is close to the beam support significantly reduces the difference between the calculated transformation factors with the ones obtained through FEA. This is expected since ratio of the shear deflection to the total deflection of the beam is inversely related to the distance between the load and the beam support. Additionally, these tables indicate that the difference between the mass transformation factor K_M based on the theoretical shape function and FEA are in general more than the corresponding difference in the load transformation factor K_L. The reason is that the mass transformation factor is a second order function of the deflection as opposed to the load transformation function which is a linear function of the deflection. Therefore, the error in the calculation of the deflection will result in more significant error in the calculation of the mass transformation factor than the load transformation factor. The results also indicate that the error in the calculation of the transformation factors based on bending-only deflection shape function is generally less for simply supported beam than the ones for fixed beam, which is expected due to the fact that the bending deflection portion of the total deflection in a simply supported beam is more significant than for a fixed beam. Therefore, for a simply supported beam, the deflection calculated based on bending only is less erroneous than that for a fixed beam.

Table 3: Transformation factors for a simply supported beam

Method of Analysis	Boundary condition	Equivalent mass/spring location (a/L)									
		1/32		1/16		1/8		1/4		1/2	
		K_L	K_M	K_L	K_M	K_L	K_M	K_L	K_M	K_L	K_M
Detailed Finite Element Analysis (FEA)	Figure 2a	2.96	10.50	1.97	4.68	1.21	1.78	0.78	0.75	0.62	0.48
Bending w/o shear deflection (difference (%) with FEA)	Figure 2a	4.25 -(44%)	22.07 -(110%)	2.26 -(15%)	6.23 -(33%)	1.27 -(5%)	1.97 -(11%)	0.79 -(1%)	0.77 -(3%)	0.63 (0%)	0.49 -(1%)
Bending with shear deflection (difference (%) with FEA)	Figure 2a	3.48 -(18%)	14.70 -(40%)	2.05 -(4%)	5.11 -(9%)	1.22 -(1%)	1.81 -(2%)	0.78 (0%)	0.75 (0%)	0.62 (0%)	0.48 (0%)

Table 4: Transformation factors a for fixed beam

Method of Analysis	Boundary condition	Equivalent mass/spring location (a/L)									
		1/32		1/16		1/8		1/4		1/2	
		K_L	K_M	K_L	K_M	K_L	K_M	K_L	K_M	K_L	K_M
Detailed Finite Element Analysis (FEA)	Figure 2b	1.06	1.51	1.03	1.45	0.88	1.08	0.63	0.58	0.50	0.37
Bending w/o shear deflection (difference (%) with FEA)	Figure 2b	4.13 -(291%)	23.62 -(1462%)	2.13 -(107%)	6.36 -(340%)	1.14 -(30%)	1.86 -(73%)	0.67 -(5%)	0.65 -(13%)	0.50 (0%)	0.37 -(1%)
Bending with shear deflection (difference (%) with FEA)	Figure 2b	0.88 (17%)	1.00 (34%)	1.00 (3%)	1.32 (9%)	0.89 -(1%)	1.08 -(1%)	0.64 (0%)	0.58 (0%)	0.50 (0%)	0.37 (1%)

Table 5: Transformation factor for a beam fixed at one end and simply supported at the other end

Method of Analysis	Loading & boundary condition	Equivalent mass/spring location (a/L)									
		1/32		1/16		1/8		1/4		1/2	
		K_L	K_M	K_L	K_M	K_L	K_M	K_L	K_M	K_L	K_M
Detailed Finite Element Analysis (FEA)	Figure 2c	1.89	4.81	1.34	2.42	0.87	1.04	0.61	0.51	0.57	0.44
Bending w/o shear deflection (difference (%) with FEA)	Figure 2c	2.90 -(53%)	11.50 -(139%)	1.57 -(17%)	3.38 -(39%)	0.91 -(5%)	1.15 -(11%)	0.62 -(1%)	0.52 -(2%)	0.57 (0%)	0.45 -(1%)
Bending with shear deflection (difference (%) with FEA)	Figure 2c	2.27 -(20%)	7.00 -(45%)	1.40 -(5%)	2.67 -(10%)	0.88 (0%)	1.05 -(1%)	0.61 (0%)	0.51 (1%)	0.57 (0%)	0.44 (1%)

Table 5: (continued)

Method of Analysis	Boundary condition	Equivalent mass/spring location (a/L)							
		3/4		7/8		15/16		31/32	
		K_L	K_M	K_L	K_M	K_L	K_M	K_L	K_M
Detailed Finite Element Analysis (FEA)	Figure 2c	0.83	0.89	1.21	1.80	1.43	2.45	1.44	2.46
Bending w/o shear deflection (difference (%) with FEA)	Figure 2c	0.89 -(7%)	1.03 -(17%)	1.62 -(34%)	3.33 -(85%)	3.12 -(118%)	12.07 -(393%)	6.11 -(324%)	46.01 -(1770%)
Bending with shear deflection (difference (%) with FEA)	Figure 2c	0.83 (0%)	0.89 -(1%)	1.20 (0%)	1.80 (0%)	1.32 (7%)	2.10 (14%)	1.10 (24%)	1.43 (42%)

Table 6: Transformation factor for slab (aspect ratio:1)

Method of Analysis	Boundary condition	Equivalent mass/spring location (a/L)									
		1/32		1/16		1/8		1/4		1/2	
		K_L	K_M	K_L	K_M	K_L	K_M	K_L	K_M	K_L	K_M
Detailed Finite Element Analysis (FEA)	Figure 4a	4.05	24.46	2.04	6.21	1.05	1.64	0.58	0.50	0.42	0.26
	Figure 4b	0.71	0.82	0.56	0.51	0.44	0.31	0.36	0.21	0.34	0.19

CONCLUSIONS

This study provides mass and load transformation factors for beam and slabs for off-centre impact loads at various locations along the structure. For the beam elements, two different deflection shape functions, one based on bending only, and one including the shear deflection were used to calculate the transformation factors. The results were compared with the refined FEA solution. The following conclusions can be made:

1- Impact locations away from the centre are associated with load and mass factors that are different from those for impact locations at centre. For example, for a simply supported beam, mass factor for impact at quarter point is 0.75. But it is 0.48 at centre. For additional comparisons, refer to Tables 3 through 6.

2- Typically, for impact locations far from the support, the effective mass and load of an equivalent SDOF is less than the total mass of the structure. However, at locations close to the supports where the deflections are small, the equivalent mass and load can far exceed the total mass and load of the structure. Incorporation of these results without consideration of the support mass and stiffness will result in large equivalent mass, stiffness and loads. The implication is that the effects of support flexibility and mass should be incorporated properly into the equivalent SDOF system whenever possible to avoid unrealistically large equivalent parameters.

3- For beam elements, the results shown in Tables 3 through 5 indicate that the transformation factors, regardless of the type of boundary conditions, could be reasonably calculated (less than 15% error) by the deformation only shape function based on Euler Bernoulli beam formula when the load distance is more than 25% of the beam length away from the support. Additionally, the transformation factors could be reasonably calculated (less than 15% error) by the inclusion of the shear deflection when the load distance is at least 5% away from the support. However, for load distances closer than 5% of the beam length away from the support, the theoretical solution shows significant difference from the FEA results and therefore, the theoretical solution may become erroneous.

4- The transformation factors for a structure with simple support condition are larger than the corresponding factors for the same structure with fixed support condition. Additionally, the errors in the calculation of the transformation factors for a structure with simple support condition are typically less than the ones for the same structure with fixed support condition.

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